Big Ramsey degrees of homogeneous structures
part 1: the prehistory

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Winter school 2022, Hejnice

Tutorial overview

I would like to cover some old&new results in the area of big Ramsey degrees:

1. The prehistory: From Ramsey theorem to big Ramsey degrees of the order of rationals and of the countable random graph.
(Milliken tree theorem, envelopes and embedding types, optimality).
2. Big Ramsey degrees using parameter spaces: Recent results on triangle-free graphs, posets and metric spaces using the Carlson-Simpson theorem.
3. Higher arities and bigger forbidden substructures: Hypergraphs, free amalgamation classes.

Ramsey Theorem

Theorem ((Infinite) Ramsey Theorem, 1930)

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$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{\omega}{p}$ into k classes (colours) there exists $X \in \binom{\omega}{\omega}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)

Big Ramsey Degrees

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In 1970's a concept of structural Ramsey theory was introduced. A Ramsey theorem can be seen as a theorem about the class of linear orders.

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Let $\mathcal{O}$ be the class of all finite linear orders.

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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Sierpiński: not true for $|O| = 2$.

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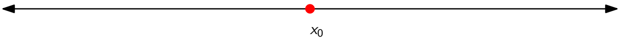

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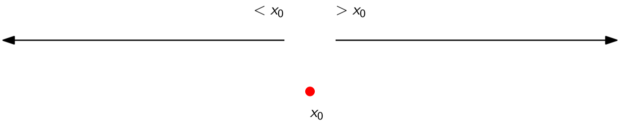

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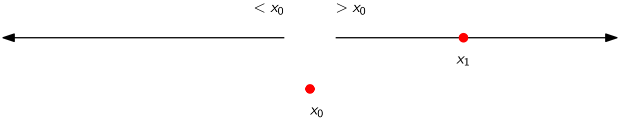

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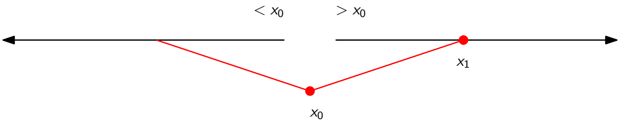

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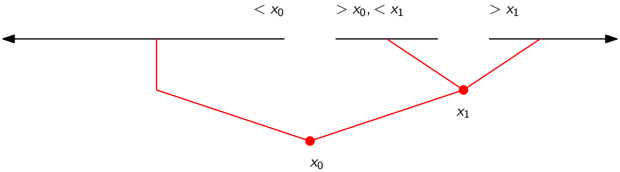

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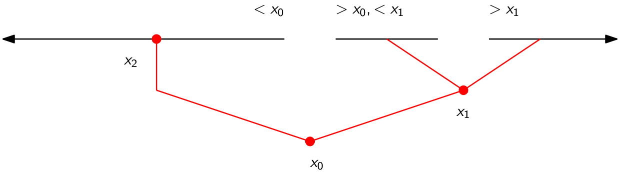

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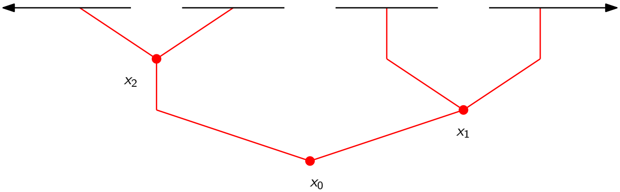

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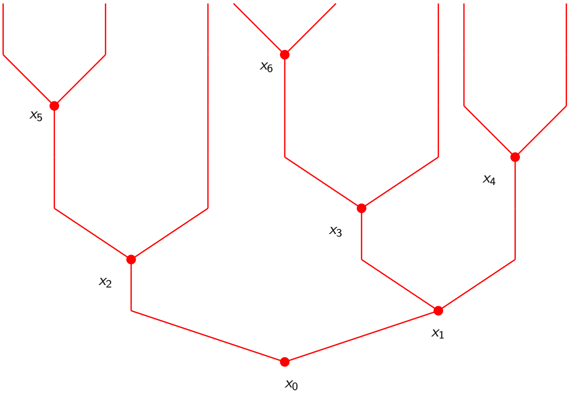

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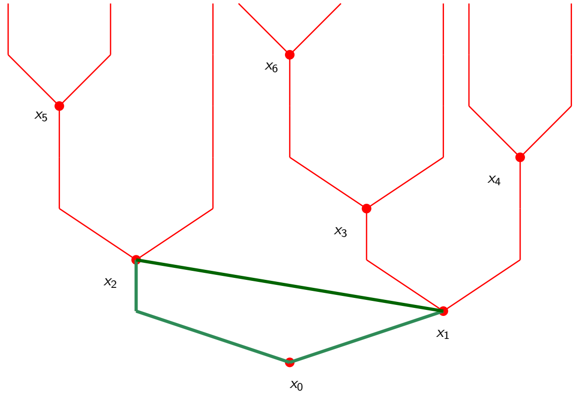

Colour of k-tuple = shape of meet closure in the tree

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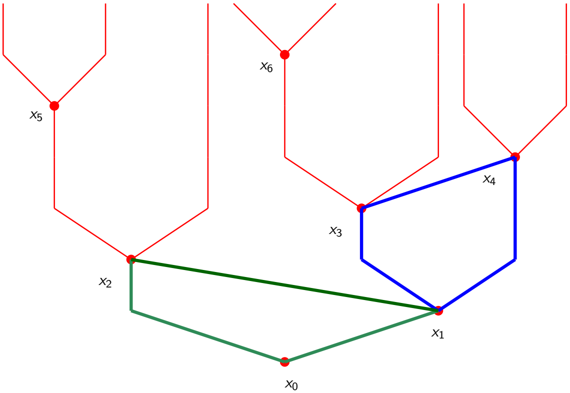

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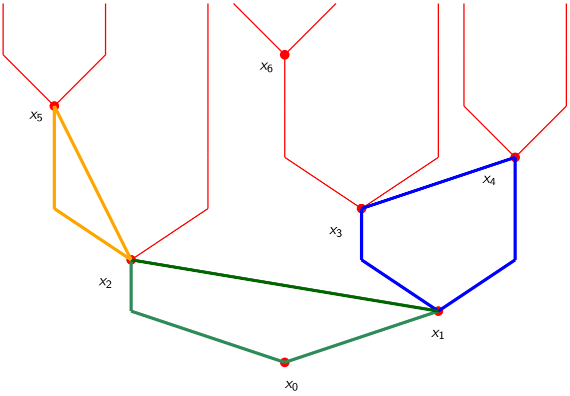

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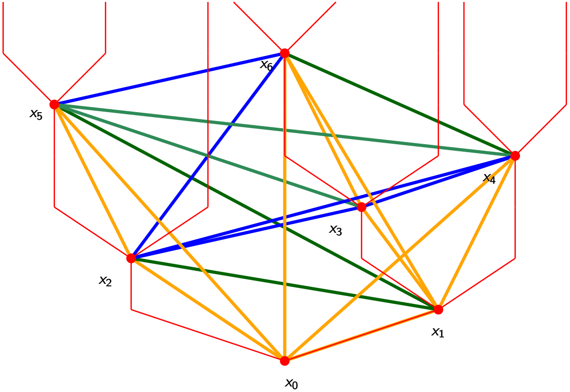

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Big Ramsey Degrees of $(\mathbb{Q}, \leq)$

In late 1960's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colours using Milliken's tree theorem.

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}} \exists_{T=T(|O|)\in\omega} \forall_{k\geq 1} : (\mathbb{Q}, \leq) \longrightarrow (\mathbb{Q}, \leq)^{(O,\leq_O)}_{k,T}.$$

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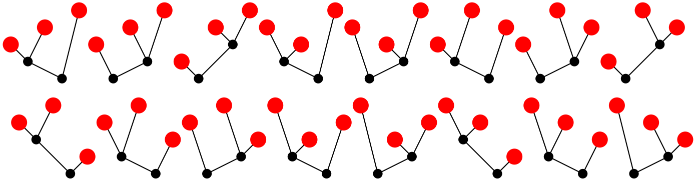
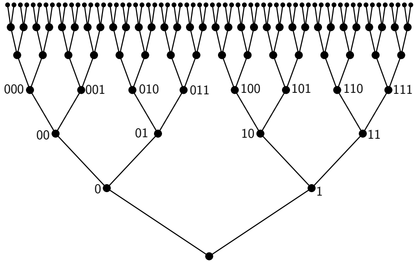


Trees (terminology)

- A tree is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every $t \in T$, the set $\{\, s \in T : s <_T t \,\}$ is finite and linearly ordered by $<_T$. All trees considered are finite or countable.
- All nonempty trees we consider are rooted, that is, they have a unique minimal element called the root of the tree.
- An element $t \in T$ of a tree T is called a node of T and its level, denoted by $|t|_T$, is the size of the set $\{\, s \in T : s <_T t \,\}$.
- We use $T(n)$ to denote the set of all nodes of T at level n,
- For $s, t \in T$, the meet $s \wedge_T t$ of s and t is the largest $s' \in T$ such that $s' \leq_T s$ and $s' \leq_T t$.
- The height of T, denoted by $h(T)$, is the minimal natural number h such that $T(h) = \emptyset$. If there is no such number h, then we say that the height of T is ω.



$2^{<\omega}$

Subtrees and strong subtrees

- A subtree of a tree T is a subset T' of T viewed as a tree equipped with the induced partial ordering such that $s \wedge_{T'} t = s \wedge_T t$ for each $s, t \in T'$.
- Given a tree T and nodes $s, t \in T$ we say that s is a successor of t in T if $t \leq_T s$.
- The node s is an immediate successor of t in T if $t <_T s$ and there is no $s' \in T$ such that $t <_T s' <_T s$.
- We denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$ and the set of immediate successors of t in T by $\mathrm{ImmSucc}_T(t)$.

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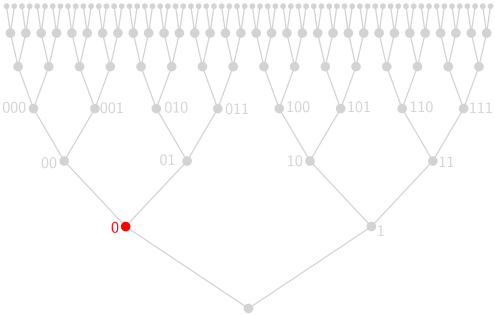
Definition (Strong subtree)

A subtree S of a tree T is a strong subtree of T if either S is empty, or S is nonempty and satisfies the following three conditions.

1. The tree S is rooted and balanced.
2. Every level of S is a subset of some level of T, that is, for every $n < h(S)$ there exists $m \in \omega$ such that $S(n) \subseteq T(m)$.
3. For every non-maximal node $s \in S$ and every $t \in \mathrm{ImmSucc}_T(s)$ the set $\mathrm{ImmSucc}_S(s) \cap \mathrm{Succ}_T(t)$ is a singleton.

Strong subtree



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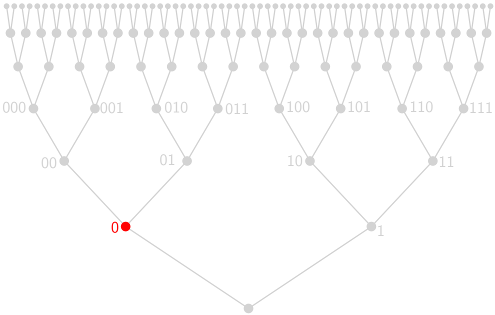

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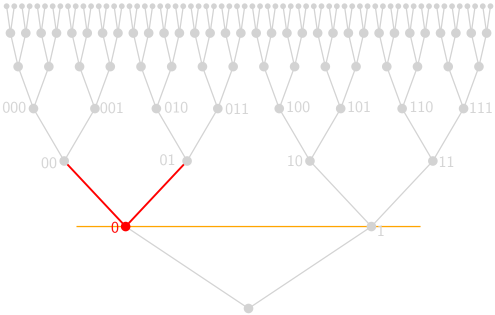

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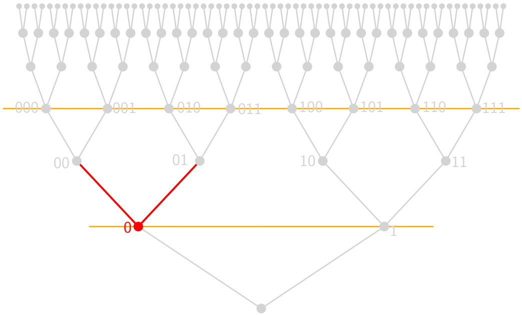

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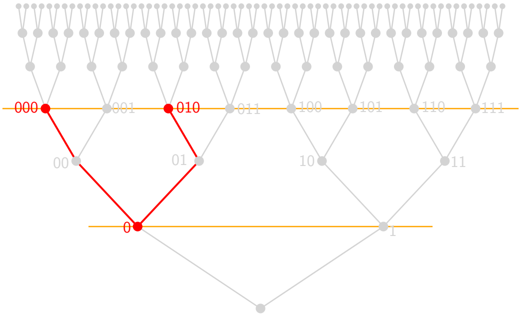

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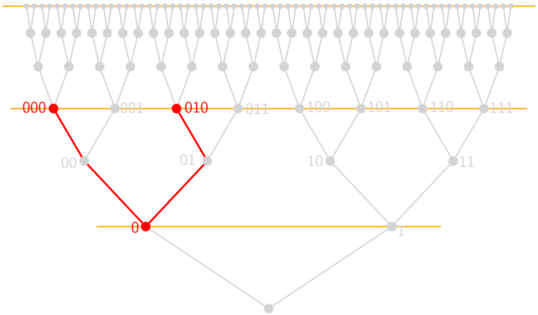

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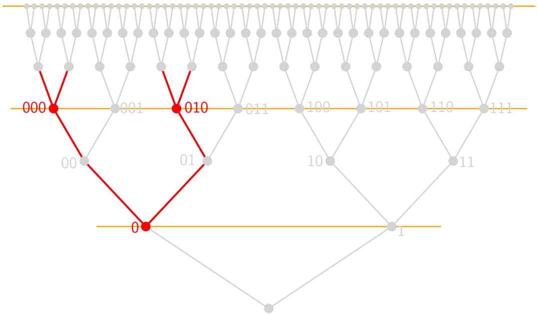

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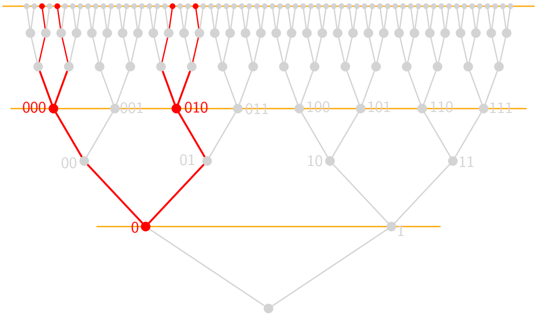

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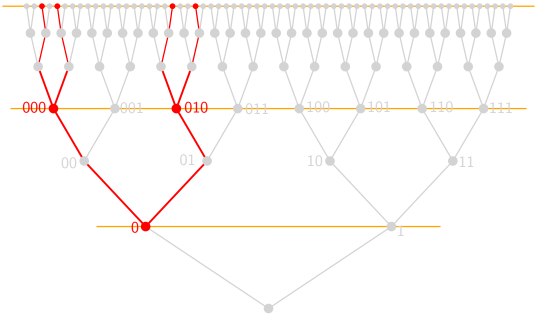

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3. Every level of S is a subset of some level of T.
4. S either has no leaves or all are at the same level.

Ramsey-type theorem for strong subtrees

Let T be a tree and $k \in \omega + 1$. We use $\mathrm{Str}_k(T)$ to denote the set of all strong subtrees of T of height k.

Theorem (Milliken 1979)

For every rooted, balanced and finitely branching tree T of infinite height, every $k \in \omega$ and every finite colouring of $\mathrm{Str}_k(T)$ there is $S \in \mathrm{Str}_\omega(T)$ such that the set $\mathrm{Str}_k(S)$ is monochromatic.

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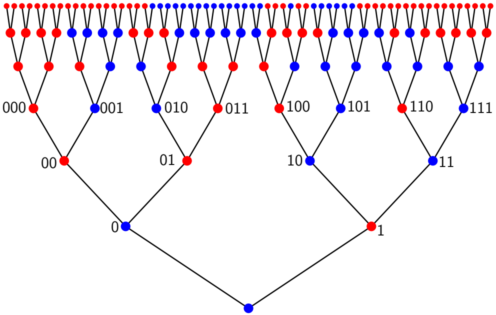

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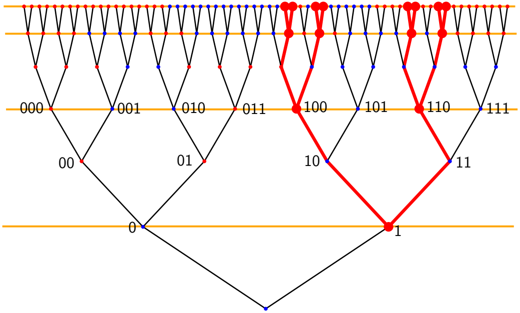

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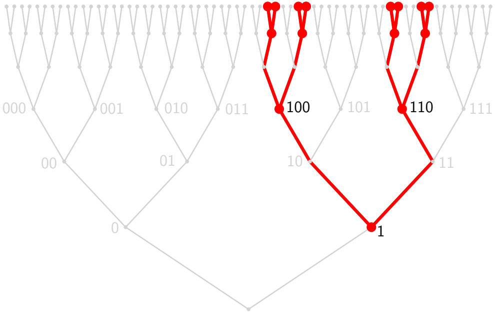

Notice that for regularly branching tree the strong subtree is isomorphic to the original tree.

Big Ramsey degrees using Milliken tree theorem

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Theorem (Laver, late 1969)

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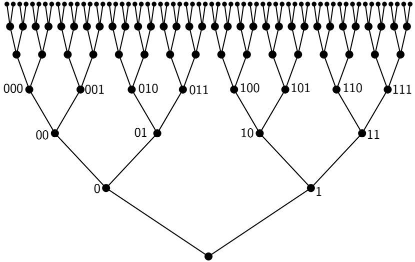

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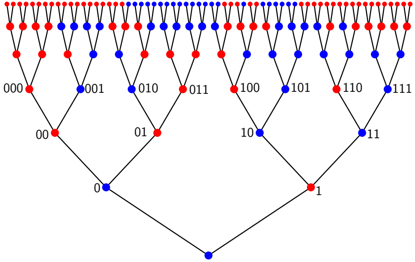

Big Ramsey degrees using Milliken tree theorem

We aim to prove:

Theorem (Laver, late 1969)

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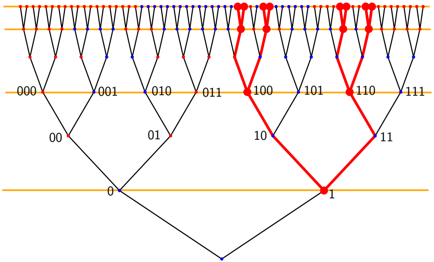

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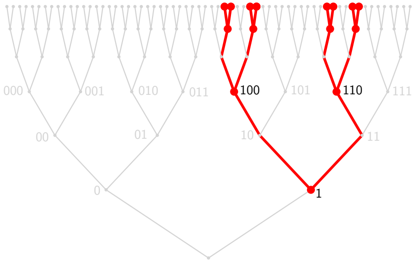

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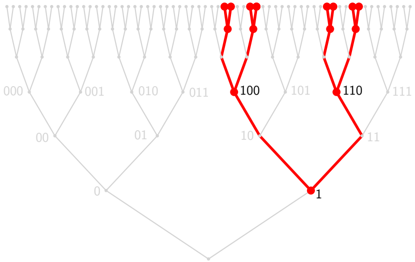
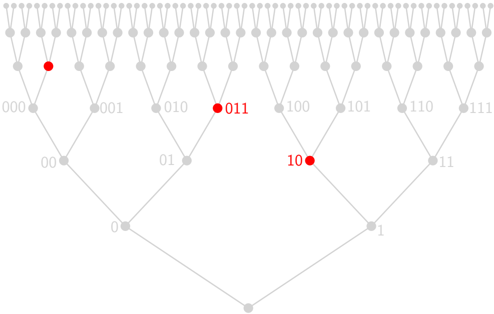


If $|O| = n > 1$ we transfer colourings of n-tuples of nodes to colouring of strong subtrees.

Envelopes of subsets



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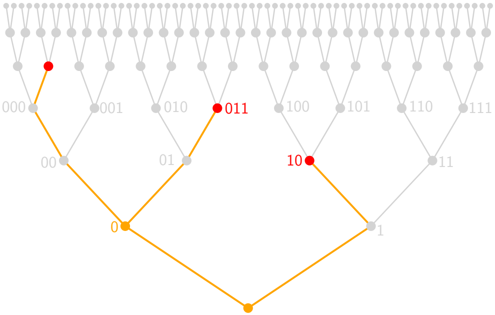

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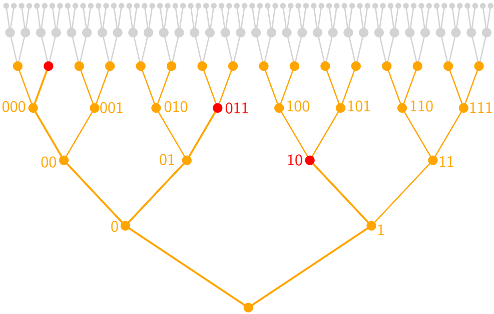

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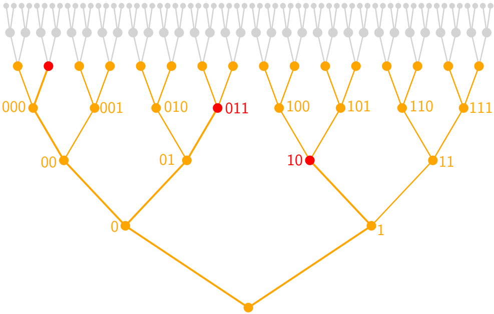

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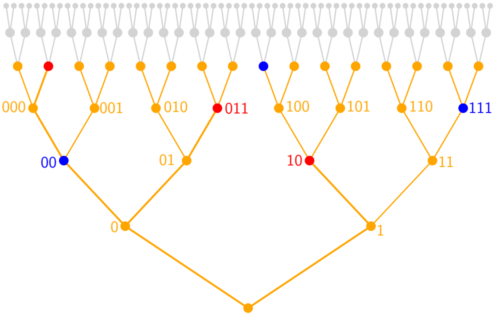

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A minimal envelope of every finite X has height at most $2|X| - 1$.

Multiple choices of X may lead to a same envelope. We speak of different embedding types within a given envelope.

Big Ramsey degrees using Milliken tree theorem

Now we can finish proof of:

Theorem (Laver, late 1969)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1}:(\mathbb{Q},\leq)\longrightarrow(\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

Proof.

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4. For each embedding type construct colouring of envelopes and pass to a monochromatic subtree by the application of Milliken tree theorem.
5. The resulting copy will have at most $T(n)$ different colours

□

Big Ramsey degrees of $(\mathbb{Q}, \leq)$ are finite!

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Definition (Devlin types)

$A \subseteq 2^{<\omega}$ a Devlin embedding type iff it is an antichain and for every $0 \leq \ell < \max_{a \in A} |a|$ precisely one of the following happens:

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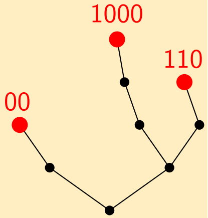

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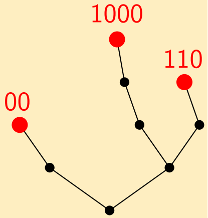

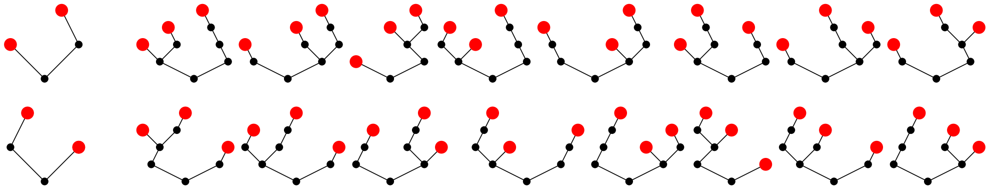

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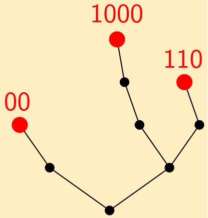

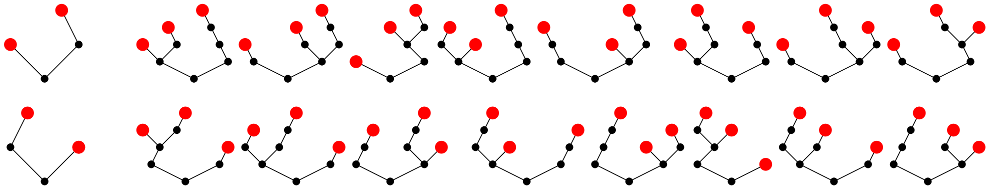

Fun fact: Number of Devlin types of size n is

$$t_n = \sum_{\ell=1}^{n-1} \binom{2n-2}{2\ell-1} t_\ell \cdot t_{n-1} \text{ with } n_1 = 1$$

This is well known sequence (of the odd tangent numbers).

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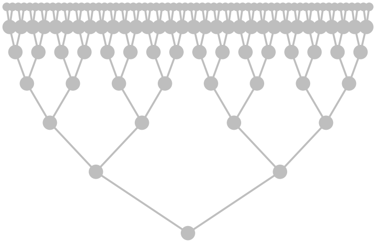

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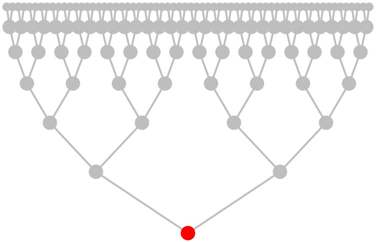

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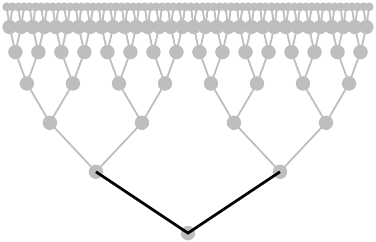

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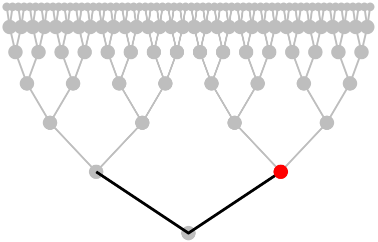

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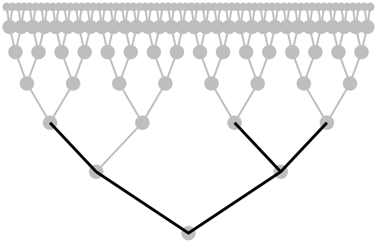

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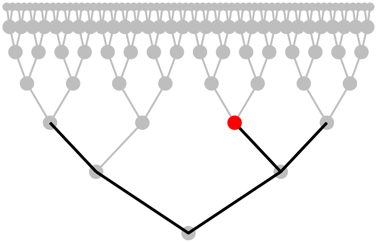

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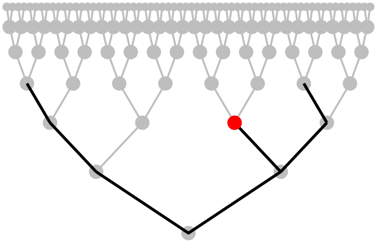

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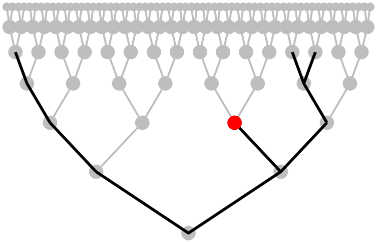

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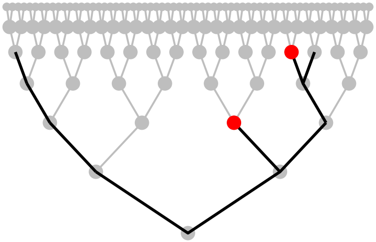

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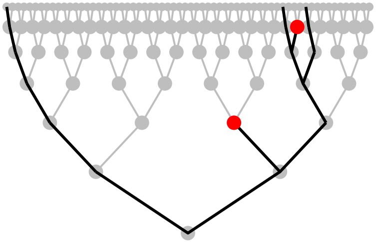

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We thus obtain an upper bound: $T(|O|)$ is at most the number of Devlin types.

The lower bound

Lemma

Let A be a Devlin type representing $(\mathbb{Q}, \leq)$ *and* $B \subseteq A$ *a copy of* $(\mathbb{Q}, \leq)$ *then there exists a* $C \subseteq B$ *whose embedding type (inside a minimal envelope) is A.*

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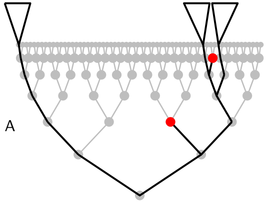
Proof by a slide-show.

□



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Proceed by induction on the individual branching/leaf events of A.

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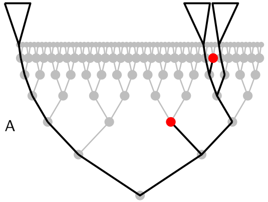

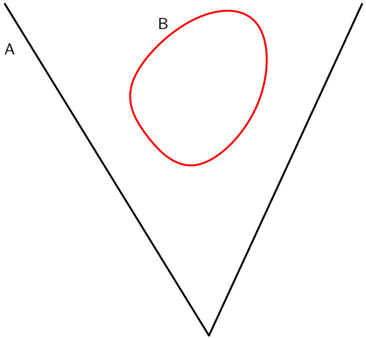

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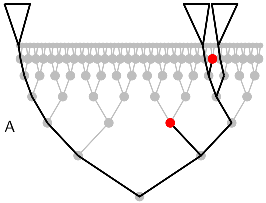
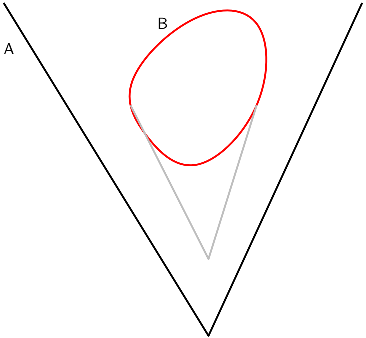
Proceed by induction on the individual branching/leaf events of A.
First produce meet of B. A starts with a branching. Place the root of A to this meet. □





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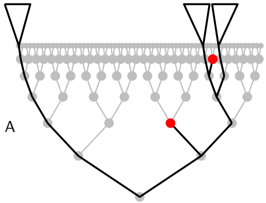

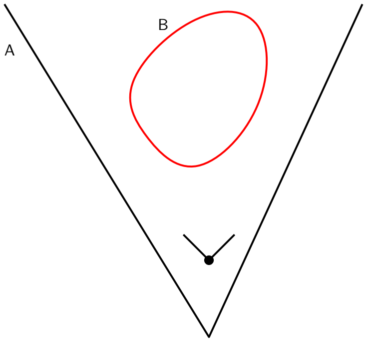

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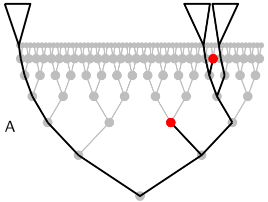
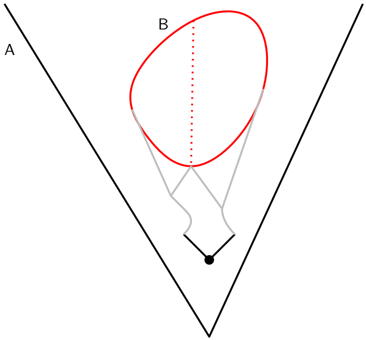
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of *A*.
The meet splits leafs of *B* into two infinite intervals. Each with meet above its son. □





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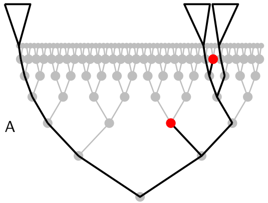
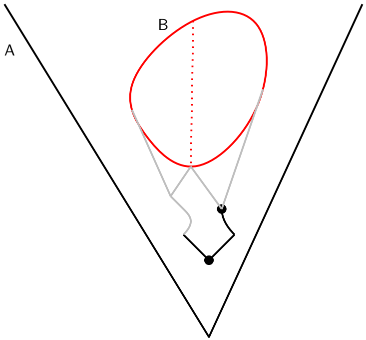
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of A.
Use corresponding meet to realize 2nd branching of A. □





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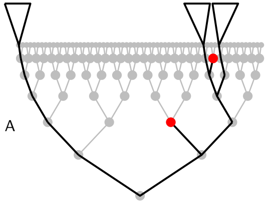
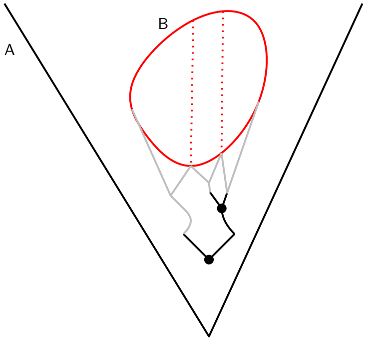
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of *A*.
B further subdivides into intervals. □





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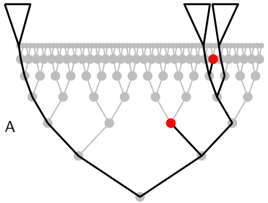
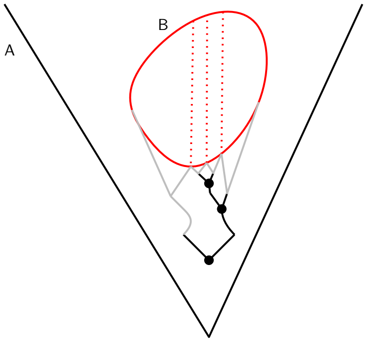
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of A.
Continue analogously with next branching. □





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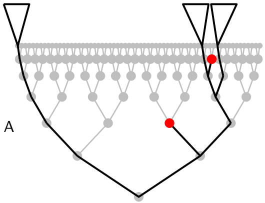
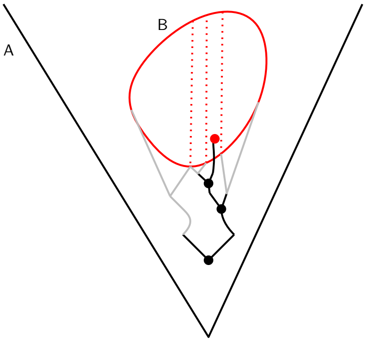
Proof by a slide-show.

Proceed by induction on the individual branching/leaf events of *A*.
Place the first leaf of *A* into corresponding interval. □





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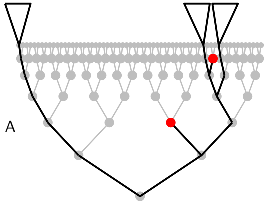
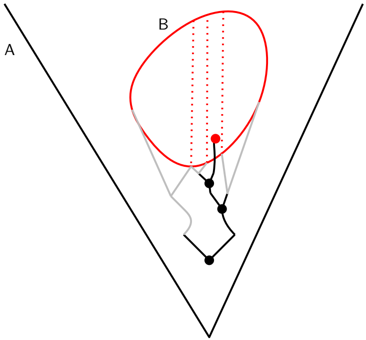
Proceed by induction on the individual branching/leaf events of A.

...blablabla...proof done. □





Victory!

We characterised the big Ramsey degrees of rationals and gave a closed-form formula. This shows that the upper bound proof is best possible. It also has applications to topological dynamics

Zucker: Big Ramsey degrees and topological dynamics

Big Ramsey degrees of **R**

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- We denote by $\mathcal{G}$ the class of all finite graphs.

Theorem

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{T=T'(\mathbf{A})\in\omega}\forall_{k\geq 1} : \mathbf{R} \longrightarrow (\mathbf{R})^{\mathbf{A}}_{k,T}.$$

This theorem was published by Sauer in 2006 and also appears in Todorčević' Introduction to Ramsey spaces. Values of $T'(\mathbf{G})$ were characterised by Laflamme–Sauer–Vuksanović in 2010.

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A finitary version is (probably more) famous!

Theorem (Nešetřil–Rödl 1977, Abramson–Harington 1978)

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{t=t(\mathbf{A})\in\omega}\forall_{\mathbf{B}\in\mathcal{G},k\geq 1}\exists_{\mathbf{C}} \in \mathcal{G} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}.$$

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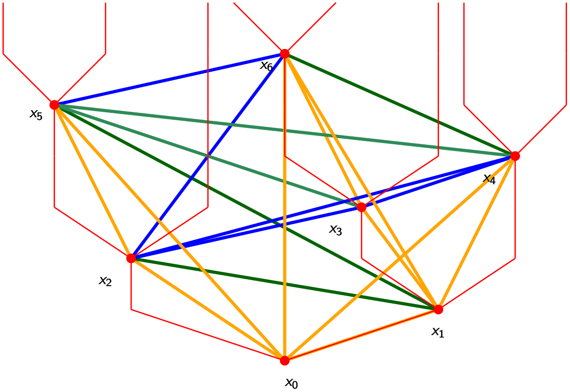
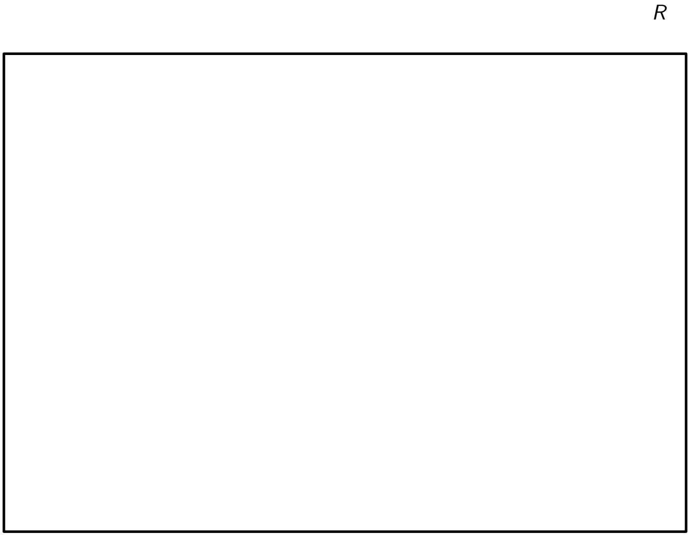


For $(\mathbb{Q}, \leq)$ we have the Sierpiński colourings. Can we do something similar for the Rado graph?

Understanding the unavoidable colourings of Rado graphs



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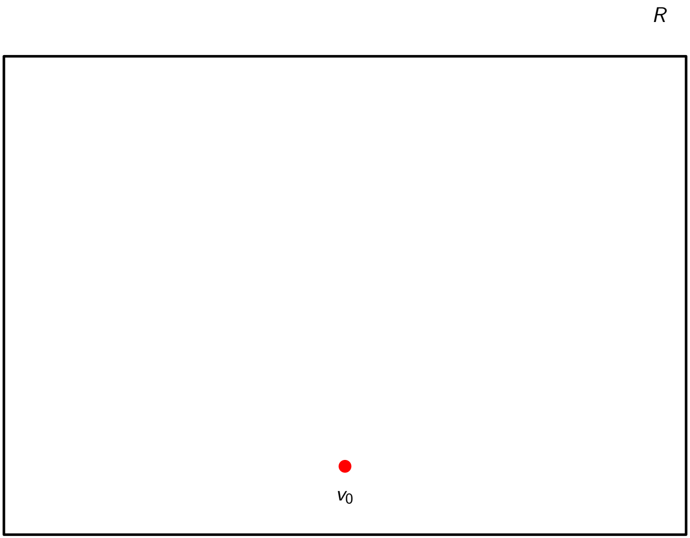

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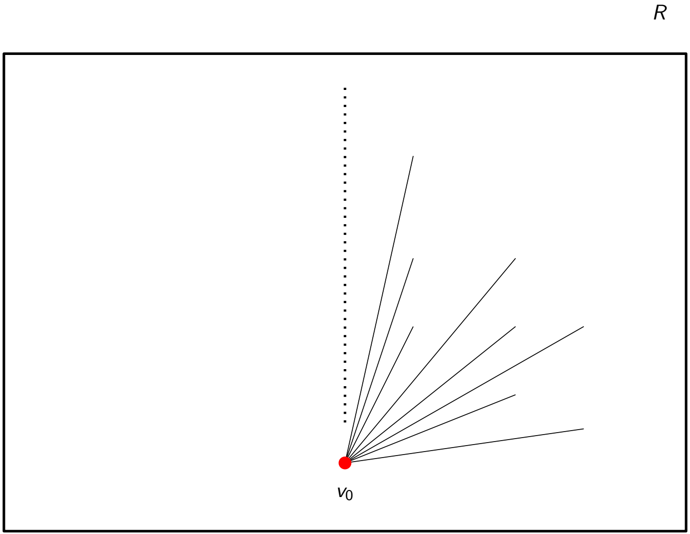

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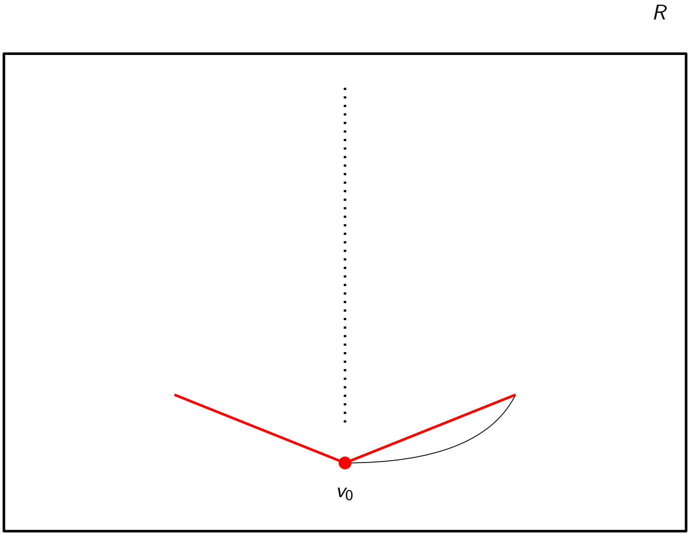

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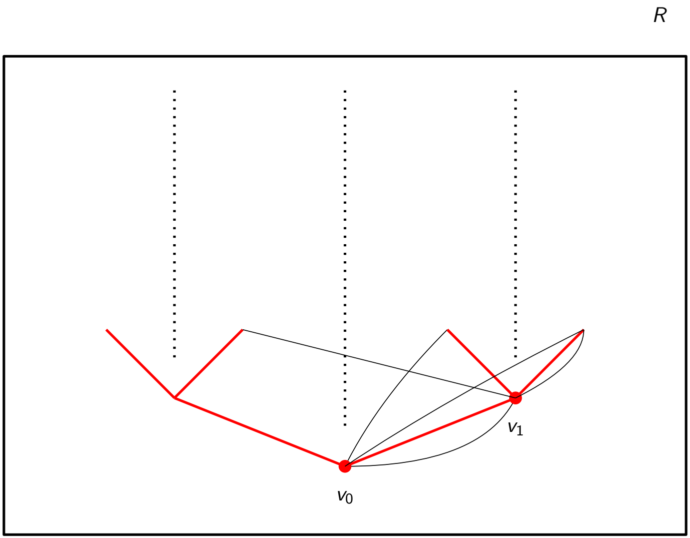

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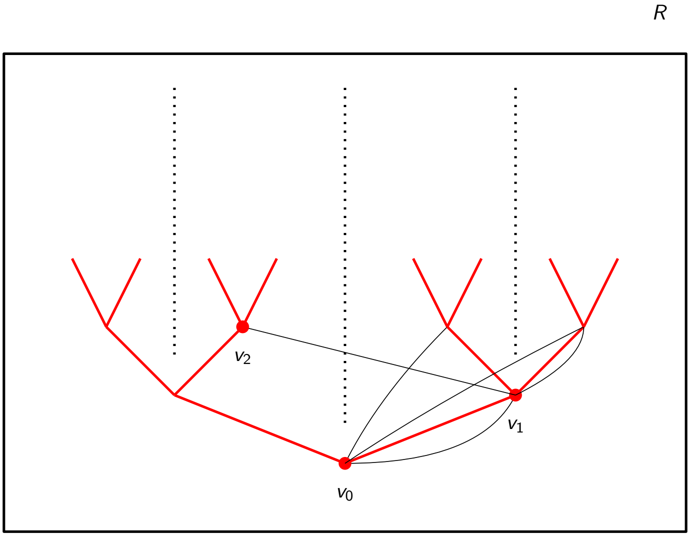

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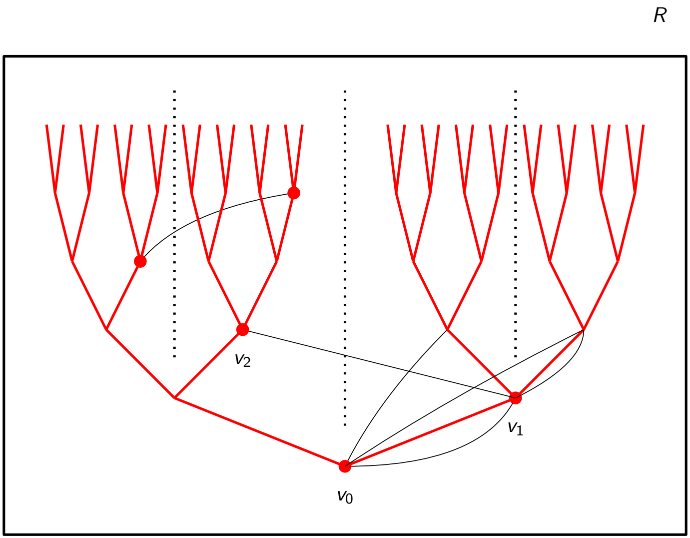

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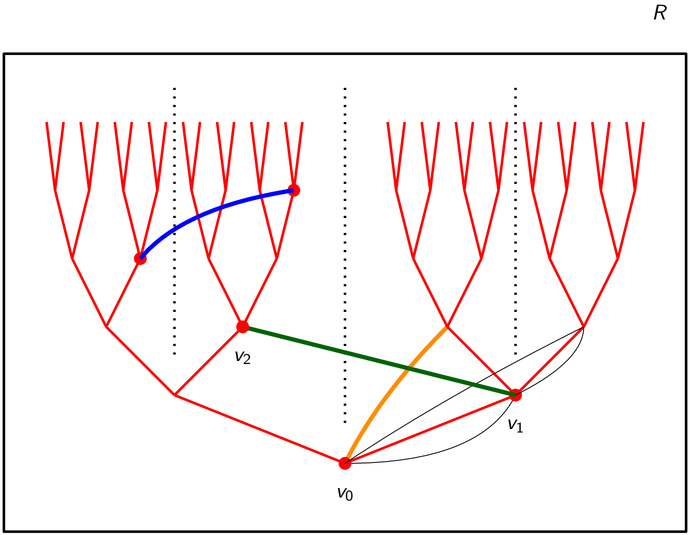
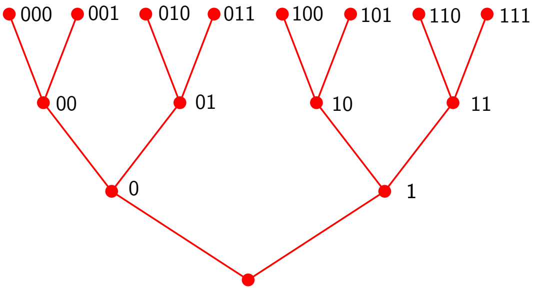


Passing number graph

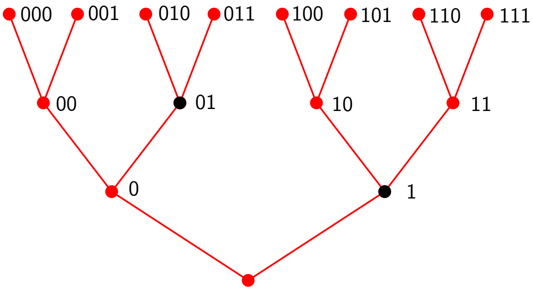


Definition (Graph G)

We will consider graph **G**:

1. Vertices: $2^{<\omega}$
2. Vertices $a, b \in 2^{<\omega}$ satisfying $|a| < |b|$ forms and edge if and only if $b(|a|) = 1$.
3. There are no other edges.

Passing number graph



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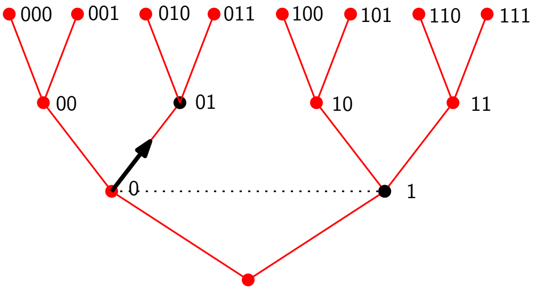

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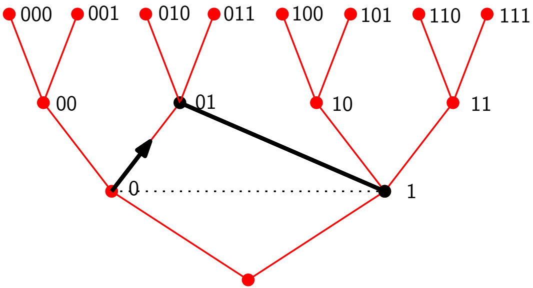

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Lemma

G *is universal: the Rado graph* **R** *embeds to* **G***.*

Proof.

Assume that the vertex set of **R** is ω. The vertex $i \in \omega$ then corresponds to a sequence a of length i with $a(j) = 1$ if and only if $i \sim j$. □

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We thus can repeat precisely the same proof as before to obtain the upper bound on big Ramsey degrees.

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Lower bounds needs a bit more care.

Thank you for the attention

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(See also S. Todorčević, Introduction to Ramsey spaces.)